Interval greedoids and families of local maximum stable sets of graphs

Vadim E. Levit
Department of Computer Science and Mathematics
Ariel University Center of Samaria, ISRAEL
levitv@ariel.ac.il

Eugen Mandrescu
Department of Computer Science
Holon Institute of Technology, ISRAEL
eugen_m@hit.ac.il

Abstract

A *maximum stable set* in a graph G is a stable set of maximum cardinality. S is a *local maximum stable set* of G, and we write $S \in \Psi(G)$, if S is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of S.

Nemhauser and Trotter Jr. [21], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of G. In [14] we have shown that the family $\Psi(T)$ of a forest T forms a greedoid on its vertex set. The cases where G is bipartite, triangle-free, well-covered, while $\Psi(G)$ is a greedoid, were analyzed in [15], [16], [18], respectively.

In this paper we demonstrate that if the family $\Psi(G)$ of the graph G satisfies the accessibility property, then $\Psi(G)$ forms an interval greedoid on its vertex set. We also characterize those graphs whose families of local maximum stable sets are either antimatroids or matroids.

Keywords: tree, bipartite graph, triangle-free graph, König-Egerváry graph, well-covered graph, simplicial graph, matroid, antimatroid.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. K_n, C_n, P_n denote respectively, the complete graph on $n \geq 1$ vertices, the chordless cycle on $n \geq 3$ vertices, and the chordless path on $n \geq 2$ vertices. The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V$ *and* $vw \in E\}$. For $A \subset V$, we denote

$$N_G(A) = \{v \in V - A : N(v) \cap A \neq \varnothing\}$$

and $N_G[A] = A \cup N(A)$, or shortly, $N(A)$ and $N[A]$, if no ambiguity.

If $|N(v)| = 1$, then v is a *pendant vertex* of G; $\mathrm{pend}(G)$ is the set of all pendant vertices of G, and by $\mathrm{isol}(G)$ we mean the set of all isolated vertices of G. If $N[v]$ is a clique, i.e., $G[N[v]]$ a complete subgraph in G, then v is a *simplicial vertex* of G, and $\mathrm{simp}(G)$ denotes the set $\{v : v \in V(G)$ *and* v *is simplicial in* $G\}$. A graph G is called *simplicial* if every vertex of G is a simplicial vertex or is adjacent to a simplicial vertex of G. A *simplex* of G is a maximal clique containing at least a simplicial vertex. The simplicial graphs were introduced by Cheston *et al.*, in [3].

Theorem 1.1 *[3] If G is a simplicial graph and $Q_1, ..., Q_s$ are its simplices, then*

$$V(G) = V(Q_1) \cup V(Q_2) \cup ... \cup V(Q_s) \text{ and } s = \theta(G) = \alpha(G),$$

where $\theta(G)$ is the minimum number of cliques that cover $V(G)$.

A *stable* set in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a *maximum stable set* of G, and the *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G. Let $\Omega(G)$ stand for the set of all maximum stable sets of G.

The following characterization of a maximum stable set of a graph, due to Berge, will be used in the sequel.

Theorem 1.2 *[1] A stable set S belongs to $\Omega(G)$ if and only if every stable set of G, disjoint from S, can be matched into S.*

A set $A \subseteq V(G)$ is a *local maximum stable set* of G if $A \in \Omega(G[N[A]])$, [14]; by $\Psi(G)$ we denote the set of all local maximum stable sets of the graph G. For instance, any stable set $S \subseteq \mathrm{simp}(G)$ belongs to $\Psi(G)$, while the converse is not generally true; e.g., $\{a\}, \{e, d\} \in \Psi(G)$ and $\{e, d\} \cap \mathrm{simp}(G) = \varnothing$, where G is the graph in Figure 1.

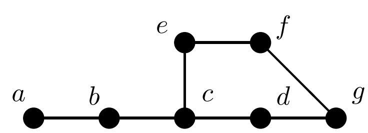

Figure 1: A graph with diverse local maximum stable sets.

The following theorem concerning maximum stable sets in general graphs, due to Nemhauser and Trotter Jr. [21], shows that for a special subgraph H of a graph G, some maximum stable set of H can be enlarged to a maximum stable set of G.

Theorem 1.3 *[21] Every local maximum stable set of a graph is a subset of a maximum stable set.*

Let us notice that the converse of Theorem 1.3 is not generally true. For instance, C_n has no proper local maximum stable set, for any $n \geq 4$. The graph G in Figure 1 shows another counterexample: any $S \in \Omega(G)$ contains some local maximum stable set, but these local maximum stable sets are of different cardinalities. As examples, $\{a, d, f\} \in \Omega(G)$ and $\{a\}, \{d, f\} \in \Psi(G)$, while for $\{b, e, g\} \in \Omega(G)$ only $\{e, g\} \in \Psi(G)$.

Definition 1.4 *[2], [10] A greedoid is a pair* $(V, \mathcal{F})$*, where* $\mathcal{F} \subseteq 2^V$ *is a non-empty set system satisfying the following conditions:*

Accessibility: *for every non-empty* $X \in \mathcal{F}$ *there is an* $x \in X$ *such that* $X - \{x\} \in \mathcal{F}$*;*

Exchange: *for* $X, Y \in \mathcal{F}, |X| = |Y| + 1$*, there is an* $x \in X - Y$ *such that* $Y \cup \{x\} \in \mathcal{F}$*.*

It is worth observing that if $(V, \mathcal{F})$ has the accessibility property and $S \in \mathcal{F}$, $|S| = k \geq 2$, then there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, ..., x_{k-1}\} \subset \{x_1, ..., x_{k-1}, x_k\} = S$$

such that $\{x_1, x_2, ..., x_j\} \in \mathcal{F}$, for all $j \in \{1, ..., k-1\}$. Such a chain we call an *accessibility chain* of S.

In the sequel we use $\mathcal{F}$ instead of $(V, \mathcal{F})$, as the ground set V will be, usually, the vertex set of some graph.

Theorem 1.5 *[14] The family of local maximum stable sets of a forest forms a greedoid on its vertex set.*

Theorem 1.5 is not specific for forests. For instance, the family $\Psi(G)$ of the graph G in Figure 2 is a greedoid.

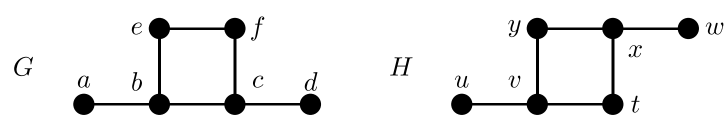

Figure 2: Both G and H are bipartite, but only $\Psi(G)$ forms a greedoid.

Notice that $\Psi(H)$ is not a greedoid, where H is from Figure 2, because the accessibility property is not satisfied, e.g., $\{y, t\} \in \Psi(H)$, but $\{y\}, \{t\} \notin \Psi(H)$.

A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of M share a common vertex. A *maximum matching* is a matching of maximum size, denoted by $\mu(G)$. A matching is *perfect* if it saturates all the vertices of the graph. A matching

$$M = \{a_i b_i : a_i, b_i \in V(G), 1 \leq i \leq k\}$$

of a graph G is called *a uniquely restricted matching* if M is the unique perfect matching of $G[\{a_i, b_i : 1 \leq i \leq k\}]$, [7]. For instance, all the maximum matchings of the graph G in Figure 2 are uniquely restricted, while the graph H from the same figure has both uniquely restricted maximum matchings (e.g., $\{uv, xw\}$) and non-uniquely restricted maximum matchings (e.g., $\{xy, tv\}$). It turns out that this is the reason that $\Psi(H)$ is not a greedoid, while $\Psi(G)$ is a greedoid.

Theorem 1.6 *[15] For a bipartite graph* G*,* $\Psi(G)$ *is a greedoid on its vertex set if and only if all its maximum matchings are uniquely restricted.*

The case of bipartite graphs owning a unique cycle, whose family of local maximum stable sets forms a greedoid is analyzed in [13].

Let us recall that G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$, [4], [24]. As a well-known example, any bipartite graph is a König-Egerváry graph, [5], [11].

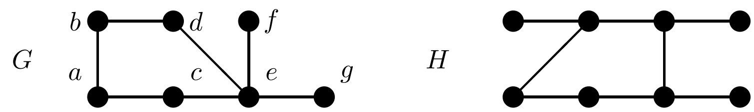

Figure 3: $\Psi(G)$ is not a greedoid, $\Psi(H)$ is a greedoid.

The graphs from Figure 3 are non-bipartite König-Egerváry graphs, and all their maximum matchings are uniquely restricted. Let us remark that both graphs are also triangle-free, but only $\Psi(H)$ is a greedoid. It is clear that $\{b, c\} \in \Psi(G)$, while $G[N[\{b, c\}]]$ is not a König-Egerváry graph. As one can see from the following theorem, this observation is the real reason for $\Psi(G)$ not to be a greedoid.

Theorem 1.7 *[16] If G is a triangle-free graph, then the following assertions are equivalent:*
(i) $\Psi(G)$ *is a greedoid;*
(ii) *all maximum matchings of G are uniquely restricted and the closed neighborhood of every local maximum stable set of G induces a König-Egerváry graph.*

Various cases of well-covered graphs whose families of local maximum stable sets form greedoids, were treated in [17], [18], [19], [20].

Let X be a graph with $V(X) = \{v_i : 1 \leq i \leq n\}$, and $\{H_i : 1 \leq i \leq n\}$ be a family of graphs. Joining each $v_i \in V(X)$ to all the vertices of H_i, we obtain a new graph, called the *corona* of X and $\{H_i : 1 \leq i \leq n\}$ and denoted by $G = X \circ \{H_1, H_2, ..., H_n\}$. For instance, see Figure 4. If $H_1 = H_2 = ... = H_n = H$, we write $G = X \circ H$, and in this case, G is called the *corona* of X and H.

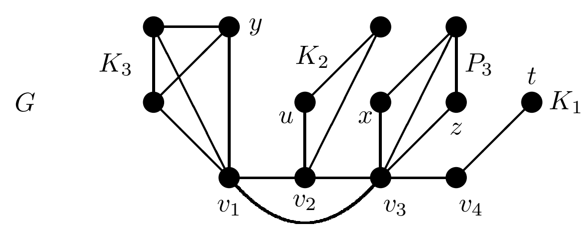

Figure 4: $G = (G[\{v_1, v_2, v_3, v_4\}]) \circ \{K_3, K_2, P_3, K_1\}$ is a well-covered graph.

Theorem 1.8 *[20] If $G = X \circ \{H_1, H_2, ..., H_n\}$ and $H_1, H_2, ..., H_n$ are non-empty graphs, then $\Psi(G)$ is a greedoid if and only if every $\Psi(H_i), i = 1, 2, ..., n$, is a greedoid.*

If each H_i is a complete graph, then $X \circ \{H_1, H_2, ..., H_n\}$ is called the *clique corona* of X and $\{H_1, H_2, ..., H_n\}$; notice that the clique corona is well-covered graph (and very well-covered, whenever $H_i = K_1, 1 \leq i \leq n$). Recall that G is *well-covered* if all its maximal stable sets have the same cardinality, [22], and G is *very well-covered* if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$, [6].

Corollary 1.9 *[18], [19] If G is the clique corona of X and $\{H_1, H_2, ..., H_n\}$, then $\Psi(G)$ is a greedoid, for any graph X.*

In this paper we show that for any graph G, the family $\Psi(G)$ satisfies the accessibility property if and only if $\Psi(G)$ is an interval greedoid. We also prove that: $\Psi(G)$ is an antimatroid if and only if G is a unique maximum stable set whose $\Psi(G)$ satisfies the accessibility property, and $\Psi(G)$ forms a matroid if and only if G is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.

2 Separating examples

Let us recall definitions of some classes of greedoids, [2].

A *matroid* is a greedoid $(V, \mathcal{F})$ enjoying the *hereditary property*:

$$\textit{if } X \in \mathcal{F} \textit{ and } Y \subset X, \textit{ then } Y \in \mathcal{F}.$$

An *antimatroid* is a greedoid $(V, \mathcal{F})$ *closed under union*:

$$\textit{if } X, Y \in \mathcal{F}, \textit{ then } X \cup Y \in \mathcal{F}.$$

A *trimmed matroid* is the intersection of a matroid and an antimatroid.

An *interval greedoid* is a greedoid $(V, \mathcal{F})$ satisfying the following condition:

$$\textit{for every } X \in \mathcal{F} \textit{ the family } \{Y \in \mathcal{F} : Y \subseteq X\} \textit{ is an antimatroid.}$$

A *local poset greedoid* is a greedoid $(V, \mathcal{F})$ satisfying the property:

$$\textit{if } X, Y, Z \in \mathcal{F} \textit{ and } X, Y \subset Z, \textit{ then } X \cup Y, X \cap Y \in \mathcal{F}.$$

The following result helps us to emphasize a number of separating examples.

Lemma 2.1 *If $\Omega(G) = \{S\}$, then $S - \{x\} \in \Psi(G)$ holds for any $x \in S$.*

Proof. Let us suppose that $S - \{x\} \notin \Psi(G)$ is true for some $x \in S$. It follows that there exists $A \in \Omega(G[N[S - \{x\}]])$ with $|A| > |S - \{x\}| = \alpha(G) - 1$. Hence, we obtain that $A = S$ which implies $x \in N(S - \{x\})$, in contradiction with the fact that $x \in S$. ■

Let us remark that Lemma 2.1 is not necessarily true when two or more vertices are deleted from the unique maximum stable set; e.g., if $\Omega(P_{2k+1}) = \{S\}$, then $\mathrm{pend}(P_{2k+1}) \subseteq S$, while $S - \mathrm{pend}(P_{2k+1}) \notin \Psi(P_{2k+1})$, for any $k \geq 2$.

- Let us observe that

  $$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$$

  is a greedoid on $\{a, b, c\}$, but there is no graph G such that $\Psi(G) = \mathcal{F}$, because, according to Lemma 2.1, $\{a, b, c\} \in \mathcal{F}$ implies that $\{b, c\} \in \mathcal{F}$, as well.

- Let us notice that

  $$\mathcal{F} = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$$

  is an antimatroid on $\{a, b, c, d\}$, but there is no graph G such that $\Psi(G) = \mathcal{F}$, because, according to Lemma 2.1, $\{a, b, c, d\} \in \mathcal{F}$ implies that $\{a, b, d\} \in \mathcal{F}$, too. Consequently, we infer also that there is an interval greedoid $\mathcal{F}$, such that $\mathcal{F} \neq \Psi(G)$ is true for any graph G.

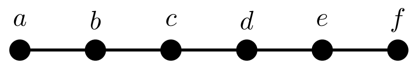

Figure 5: A tree T whose $\Psi(T)$ is neither a matroid nor an antimatroid.

- If $G = \overline{K_n}$, then $\Psi(G)$ produces both a matroid, an antimatroid and a local poset greedoid. The same is true for some trees, e.g., for P_3.
- The family of maximum local stable sets of the tree P_6 (see Figure 5) is not a matroid because while $\{a, c\} \in \Psi(P_6)$, the set $\{c\}$ does not belong to $\Psi(P_6)$. The family $\Psi(P_6)$ is not an antimatroid, too. One of the reasons is that while $\{a, c\}, \{d, f\} \in \Psi(P_6)$, the set $\{a, c\} \cup \{d, f\}$ is not even stable.
- It is also easy to check that: $\Psi(P_5)$ is an antimatroid and not a matroid; $\Psi(P_2)$ is a matroid, but it is not an antimatroid.
- If $G = P_4$ or $G = K_{1,n}, n \geq 1$, then $\Psi(G)$ is a local poset greedoid.
- $\Psi(P_5)$ is a greedoid, but it is not a local poset greedoid. To see that, let us consider $X = \{a, b\}, Y = \{b, c\}, Z = \{a, b, c\}$, that clearly satisfy

$$X, Y, Z \in \Psi(P_5), X \subset Z, Y \subset Z, X \cup Y \in \Psi(P_5),$$

but $X \cap Y = \{b\} \notin \Psi(P_5)$.

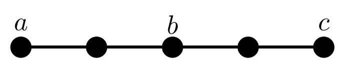

Figure 6: $\Psi(P_5)$ is a greedoid, but not a local poset greedoid.

- Let $V(P_4) = \{a, b, c, d\}, E(P_4) = \{ab, bc, cd\}$. Then, $\Psi(P_4)$ is a greedoid, but is neither a matroid, since

$$\{a, c\} \in \Psi(P_4), \text{ but } \{c\} \notin \Psi(P_4),$$

nor an antimatroid, because

$$\{a, c\}, \{b, d\} \in \Psi(P_4), \text{ while } \{a, b, c, d\} \notin \Psi(P_4).$$

On the other hand, the family

$$M = \{\varnothing, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$$

is a matroid, the family

$$AM = \{\{a\}, \{d\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}\}$$

is an antimatroid, and $\Psi(P_4) = M \cap AM$, i.e., $\Psi(P_4)$ is a trimmed matroid.

3 An interval greedoid on vertex set of a graph

Let us observe that the family $\Psi(G)$ is not generally closed under intersection or difference, even if G has a unique maximum stable set. For instance, the tree P_7 in Figure 7 has a unique maximum stable set, namely $\{a, c, e, g\}$, and while

$$A = \{a, c\}, B = \{a, d\}, C = \{c, e, g\} \in \Psi(P_7),$$

none of the sets $A - B, A \cap C$ belong to $\Psi(P_7)$.

However, if every connected component of G is a complete graph, then $\Psi(G)$ is obviously closed under intersection or difference. As far as the union operation is concerned, we have the following general statement.

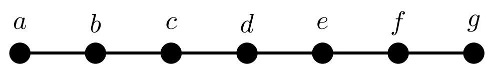

Figure 7: A tree T with a unique maximum stable set: $\{a, c, e, g\}$.

Theorem 3.1 *For any graph G, if $A, B \in \Psi(G)$ and $A \cup B$ is stable, then $A \cup B \in \Psi(G)$.*

Proof. For $S \in \Omega(N[G[A \cup B]])$ let us denote:

$$\begin{aligned} S_A &= S \cap (N[A] - N[A \cap B]), \\ S_B &= S \cap (N[B] - N[A \cap B]), \\ S_{AB} &= S \cap N[A \cap B]. \end{aligned}$$

Since $A, B \in \Psi(G)$, it follows also that

$$|S_A| + |S_{AB}| \leq |A| \text{ and } |S_B| + |S_{AB}| \leq |B| .$$

On the other hand, $|S_{AB}| \geq |A \cap B|$, because otherwise, $S_A \cup (A \cap B) \cup S_B$ is stable in $N[A \cup B]$ with $|S_A \cup (A \cap B) \cup S_B| > |S|$, in contradiction with the choice $S \in \Omega(N[G[A \cup B]])$. Consequently, we obtain:

$$|S_A| + |S_{AB}| + |S_B| + |A \cap B| \leq |S_A| + 2 |S_{AB}| + |S_B| \leq |A| + |B|$$

which implies:

$$|S| = |S_A| + |S_{AB}| + |S_B| \leq |A| + |B| - |A \cap B| = |A \cup B| .$$

Hence, we get that $A \cup B \in \Omega(G[N[A \cup B]])$, i.e., $A \cup B \in \Psi(G)$. ■

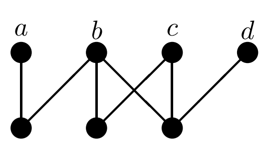

Figure 8: A graph satisfying $A \cap \mathrm{simp}(G) \neq \emptyset$ for every $A \in \Psi(G)$.

The condition "$A \cap \mathrm{simp}(G) \neq \emptyset$, for any $A \in \Psi(G)$" is clearly necessary, but is not sufficient to guarantee the accessibility property for the family $\Psi(G)$; e.g., the graph G in Figure 8 has $\{a,b,c\} \in \Psi(G), \{a,b,c\} \cap \mathrm{simp}(G) = \{a\}$, but no subset consisting of two elements of $\{a,b,c\}$ belongs to $\Psi(G)$.

It is worth observing that if $\Psi(G)$ has the accessibility property and $S \in \Psi(G)$, $|S| = k \geq 2$, then there is a chain

$$\{x_1\} \subset \{x_1, x_2\} \subset ... \subset \{x_1, ..., x_{k-1}\} \subset \{x_1, ..., x_{k-1}, x_k\} = S$$

such that $\{x_1, x_2, ..., x_j\} \in \Psi(G)$, for all $j \in \{1, ..., k-1\}$. Such a chain we call an *accessibility chain* of S.

Theorem 3.2 *If the family $\Psi(G)$ of a graph G satisfies the accessibility property, then the following assertions are true:*
(i) *$\Psi(G)$ forms a greedoid on its vertex set;*
(ii) *$\Psi(G)$ is an interval greedoid.*

Proof. *(i)* We have to prove that $\Psi(G)$ satisfies also the exchange property.

Let $A, B \in \Psi(G)$ such that $|B| = |A| + 1 = m + 1$. Hence, there is an accessibility chain for B, say

$$\{b_1\} \subset \{b_1, b_2\} \subset ... \subset \{b_1, ..., b_m\} \subset B.$$

Since B is stable, $A \in \Psi(G)$ but $|A| < |B|$, it follows that there exists some $b \in B - A$, such that $b \notin N[A]$.

If $b = b_1$, then

$$A \cup \{b_1\} \leq \alpha(N[A \cup \{b_1\}]) = \alpha(N[A] \cup N[\{b_1\}]) \leq \alpha(N[A]) + \alpha(N[\{b_1\}]) = |A| + 1 = |A \cup \{b_1\}|,$$

because b_1 is a simplicial vertex and $A \cup \{b_1\}$ is a stable set. Consequently, $A \cup \{b_1\} \in \Psi(G)$.

Otherwise, let $b_{k+1} \in B, k \geq 1$ be the first vertex in B satisfying the conditions:

$$b_1, ..., b_k \in N[A] \text{ and } b_{k+1} \notin N[A].$$

Since $\{b_1, ..., b_k\}$ is stable in $G[N[A]]$ and $A \in \Omega(G[N[A]])$, Theorem 1.2 implies that there is a matching M from $\{b_1, ..., b_k\} - A$ into A, i.e., there is $\{a_1, ..., a_k\} \subseteq A$ such that for any $i \in \{1, ..., k\}$ either $a_i = b_i$ or $a_i b_i \in M$.

We show that $A \cup \{b_{k+1}\} \in \Psi(G)$.

If not, there exists some $\{c_1, ..., c_p, d_1, ..., d_s\}$ in $\Omega(G[N[A \cup \{b_{k+1}\}]])$ such that:

$$p + s \geq m + 2, \{c_1, ..., c_p\} \subseteq N[A] \text{ and } \{d_1, ..., d_s\} \subseteq N(b_{k+1}).$$

Since $\{b_1, ..., b_{k+1}\}$ is in $\Psi(G)$, $\{a_1, ..., a_k, d_1, ..., d_s\} \subseteq N[\{b_1, ..., b_{k+1}\}]$, while $\{a_1, ..., a_k\}$ and $\{d_1, ..., d_s\}$ are stable sets, it follows that

$$|\{d_1, ..., d_s\} \cap N[\{a_1, ..., a_k\}]| \geq s - 1,$$

because otherwise $\{a_1, ..., a_k, d_1, ..., d_s\}$ contains some stable set of $k+2$ vertices, contradicting the fact that

$$\{b_1, ..., b_{k+1}\} \in \Omega(G[N[\{b_1, ..., b_{k+1}\}]]).$$

So, we may suppose that $\{d_1, ..., d_{s-1}\} \subseteq N[\{a_1, ..., a_k\}]$. Since

$$\{c_1, ..., c_p\} \subset N[A] \text{ and } \{d_1, ..., d_{s-1}\} \subseteq N[\{a_1, ..., a_k\}],$$

it follows that

$$W = \{c_1, ..., c_p, d_1, ..., d_{s-1}\} \subseteq N[A]$$

and W is a stable set of size

$$|W| = p + s - 1 \geq m + 1,$$

i.e., W is larger than A, in contradiction with the choice $A \in \Psi(G)$.

(ii) For $A \in \Psi(G)$ let us denote

$$\Psi(A) = \{B \in \Psi(G) : B \subseteq A\}.$$

Since, by part *(i)*, $\Psi(G)$ is a greedoid, it is clear that $\Psi(A)$ is also a greedoid. For any B_1, B_2 belonging to $\Psi(A)$, the set $B_1 \cup B_2$ is stable, because A is stable. According to Theorem 3.1, it follows that $B_1 \cup B_2 \in \Psi(A)$. Hence, $\Psi(A)$ is an antimatroid and consequently, $\Psi(G)$ is an interval greedoid. ■

As a consequence, we may say that all the greedoids we have obtained by Theorems 1.5, 1.6, 1.7, and 1.8, are interval greedoids.

Corollary 3.3 *The family $\Psi(G)$ of a graph G satisfies the accessibility property if and only if $\Psi(G)$ forms an interval greedoid.*

4 The graphs whose $\Psi(G)$ is either an antimatroid or a matroid

If $|\Omega(G)| = 1$, then G is called a *unique maximum stable set graph*, [8], [9], [12], [23].

Lemma 4.1 *G is a unique maximum stable set graph if and only if $\Psi(G)$ is closed under union.*

Proof. Let $\Omega(G) = \{S\}$ and $A, B \in \Psi(G)$. By Theorem 1.3, both A and B are subsets of S. Hence, $A \cup B$ is a stable set in G, and according to Theorem 3.1, we infer that $A \cup B \in \Psi(G)$.

Conversely, let $\Psi(G)$ be closed under union. If $\Omega(G)$ contains two different elements, say S_1, S_2, then $S_1, S_2 \in \Psi(G)$ and consequently, $S_1 \cup S_2 \in \Psi(G)$. Hence, $S_1 \cup S_2$ must be a stable set in G, in contradiction with $|S_1 \cup S_2| > \alpha(G)$. ■

Notice that the graphs G_1, G_2 from Figure 9 are unique maximum stable set graphs, but only $\Psi(G_1)$ does not satisfy the accessibility property, since $\{y, z\} \in \Psi(G_1)$, while $\{y\}, \{z\}$ do not belong to $\Psi(G_1)$. Hence, by Theorem 3.2, only $\Psi(G_2)$ is a greedoid. Moreover, the following theorem shows that $\Psi(G_2)$ is even an antimatroid.

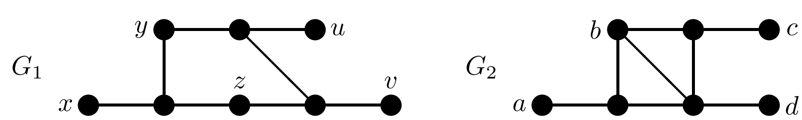

Figure 9: $\Omega(G_i) = \{S_i\}, i = 1, 2$, where $S_1 = \{x, y, z, u, v\}$ and $S_2 = \{a, b, c, d\}$.

Theorem 4.2 *For any graph G, the following assertions are equivalent:*
(i) $\Psi(G)$ *is an antimatroid;*
(ii) *G is a unique maximum stable set graph and $\Psi(G)$ satisfies the accessibility property.*

Proof. If $\Psi(G)$ is an antimatroid, then $\Psi(G)$ satisfies the accessibility property and is closed under union. By Theorem 3.1, G must be a unique maximum stable set graph.

Conversely, since $\Psi(G)$ satisfies the accessibility property, Theorem 3.2 ensures that $\Psi(G)$ is a greedoid. Further, according to Lemma 4.1, $\Psi(G)$ is also closed under union, because G is a unique maximum stable set graph. Consequently, $\Psi(G)$ is an antimatroid. ■

For instance, all the graphs from Figure 10 are unique maximum stable graphs, but only $\Psi(G_1)$ and $\Psi(G_2)$ are antimatroids; $\Psi(G_3)$ is not a greedoid, since $\{x, y\} \in \Psi(G_3)$, while $\{x\}, \{y\} \notin \Psi(G_3)$.

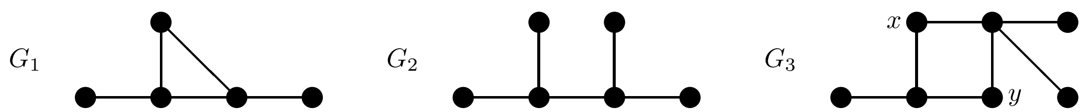

Figure 10: G_1, G_2 and G_3 are unique maximum stable graphs.

Corollary 4.3 *If T is a tree, then the following assertions are equivalent:*
(i) $\Psi(T)$ *is an antimatroid;*
(ii) *T is a unique maximum stable set graph;*
(iii) *T has a maximum stable set S such that $|N(v) \cap S| \geq 2$ holds for every $v \in V(T) - S$.*

Proof. The equivalence *(i)* $\Longleftrightarrow$ *(ii)* follows from Theorems 4.2, 1.5.
The equivalence *(ii)* $\Longleftrightarrow$ *(iii)* was proved in [8], [25]. ■
As far as the graphs in Figure 11 are concerned, it is easy to check that:

- $\Psi(G_1)$ is not a greedoid, because $\{u, v\} \in \Psi(G_1)$, but $\{a\}, \{b\} \notin \Psi(G_1)$;
- $\Psi(G_2)$ is a greedoid, but not a matroid, since $\{a, b\} \in \Psi(G_2)$, while $\{a\} \notin \Psi(G_2)$;
- $\Psi(G_3)$ is a matroid.

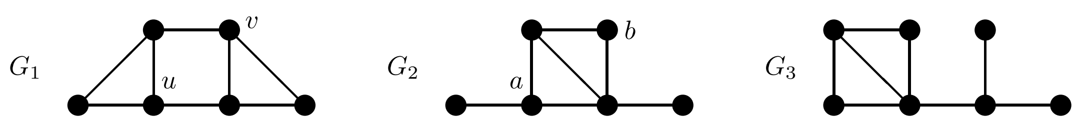

Figure 11: G_1, G_2 and G_3 are simplicial graphs.

Theorem 4.4 *If G is a graph, then the following assertions are equivalent:*
(i) $\Psi(G)$ *is a matroid;*
(ii) $S \subseteq \mathrm{simp}(G)$, *for every $S \in \Omega(G)$;*
(iii) *G is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.*

Proof. *(i) $\Rightarrow$(ii)* Suppose that $\Psi(G)$ is a matroid. Any $S \in \Omega(G)$ belongs also to $\Psi(G)$, and therefore, by hereditary property, it follows that $\{x\} \in \Psi(G)$, for every $x \in S$. Hence, $\alpha(G[N[x]]) = |\{x\}| = 1$, and this ensures that $N[x]$ is a clique. Consequently, we infer that $x \in \mathrm{simp}(G)$, for each $x \in S$. Therefore, $S \subseteq \mathrm{simp}(G)$, for every $S \in \Omega(G)$.

(ii) $\Rightarrow$ (i) According to Theorem 3.2, it is sufficient to show that $\Psi(G)$ has hereditary property.

Let now $S_1 \in \Psi(G)$ and $S_2 \subset S_1$. By Theorem 1.3, there is some $S \in \Omega(G)$ such that $S_1 \subset S$. Hence, $S_2 \subseteq \mathrm{simp}(G)$, which clearly implies that $S_2 \in \Psi(G)$.

(ii) $\Rightarrow$ (iii) Suppose that G is not simplicial. Then there is at least one vertex $v \in V(G)$ such that $N[v] \cap \mathrm{simp}(G) = \varnothing$. For each $S \in \Omega(G)$ we have $S \subseteq \mathrm{simp}(G)$, and this implies that $S \cap N[v] = \emptyset$. Hence, $S \cup \{v\}$ is stable in G, in contradiction with the choice $S \in \Omega(G)$. Therefore, G is a simplicial graph.

Assume that there exists a vertex $v \in V(G) - \mathrm{simp}(G)$ such that v belongs to a unique simplex, say Q, and let $S \in \Omega(G)$. Since $S \subseteq \mathrm{simp}(G)$ and $v \notin \mathrm{simp}(G)$, it follows that $S \cap Q = \{w\} \neq \{v\}$. Hence, we get that $(S \cup \{v\}) - \{w\} \in \Omega(G)$, and consequently, $(S \cup \{v\}) - \{w\} \subseteq \mathrm{simp}(G)$, contradicting the assumption that $v \notin \mathrm{simp}(G)$.

So, we may conclude that G is a simplicial graph and every non-simplicial vertex belongs to at least two different simplices.

(iii) $\Rightarrow$ (ii) According to Theorem 1.1,

$$V(G) = V(Q_1) \cup V(Q_2) \cup ... \cup V(Q_s),$$

where $Q_1, ..., Q_s$ are the simplices of G and $s = \theta(G) = \alpha(G)$. Suppose that there is some $S \in \Omega(G)$ such that $S \nsubseteq \mathrm{simp}(G)$. Let $v \in S - \mathrm{simp}(G)$ and Q_i, Q_j be two different simplices of G, both containing v. Since $v \in S$ and Q_i, Q_j are cliques in G, it follows that $S \cap Q_i = \{v\} = S \cap Q_j$. Let $v_i \in Q_i \cap \mathrm{simp}(G)$ and $v_j \in Q_j \cap \mathrm{simp}(G)$ be non-adjacent vertices in G. Then, the set $(S \cup \{v_i, v_j\}) - \{v\}$ is stable in G and larger than S, in contradiction with $S \in \Omega(G)$. Therefore, $S \subseteq \mathrm{simp}(G)$ must hold for each $S \in \Omega(G)$, and this completes the proof. ■

Corollary 4.5 *If G is a triangle-free graph, then the following statements are equivalent:*

(i) $\Psi(G)$ *is a matroid;*

(ii) $S \subseteq \mathrm{pend}(G) \cup \mathrm{isol}(G)$, *for every* $S \in \Omega(G)$*;*

(iii) G *has as connected components:* K_1, K_2*, and graphs having unique maximum stable sets, namely, sets of their pendant vertices.*

Proof. Now, $\mathrm{simp}(G) = \mathrm{pend}(G) \cup \mathrm{isol}(G)$, since G is a triangle-free graph. Further, the proof follows from Theorem 4.4. ■

Since bipartite graphs are triangle-free, Corollary 4.5 is true for bipartite graphs, as well. It is easy to see that $\Psi(K_1)$ and $\Psi(K_2)$ are matroids. For trees with more than three vertices, we have the following result.

Corollary 4.6 *If T is a tree of order at least three, then the following assertions are equivalent:*

(i) $\Psi(T)$ *is a matroid;*

(ii) $\mathrm{pend}(T)$ *is the unique maximum stable set of* T*;*

(iii) $\Psi(T)$ *is a trimmed matroid.*

Proof. Corollary 4.5 assures that "*(i)* $\iff$ *(ii)*" is valid. Further, using Corollary 4.3, it follows that "*(ii)* $\implies$ *(iii)*" is also true. Clearly, *(iii)* implies *(i)*. ■

If T is a tree having a unique maximum stable set, then $\Psi(T)$ is a greedoid, but is not necessarily a local poset greedoid; e.g., the tree in Figure 6.

Proposition 4.7 *If every $S \in \Omega(G)$ is contained in* $\mathrm{simp}(G)$*, then $\Psi(G)$ is a local poset greedoid.*

Proof. First, $\Psi(G)$ is a greedoid, by Theorem 4.4. Further, let us notice that if a stable set S is contained in $\mathrm{simp}(G)$, then S belongs to $\Psi(G)$. Therefore, for any $X, Y, Z \in \Psi(G)$ satisfying $X \subset Z, Y \subset Z$, it follows that $X \cup Y, X \cap Y \in \Psi(G)$. Hence, $\Psi(G)$ is a local poset greedoid. ■

Let us notice that the converse of Proposition 4.7 is not true. For instance, $\Psi(P_4)$ is a local poset greedoid, and, clearly, there exists $S \in \Omega(P_4)$, which is not contained in $\mathrm{simp}(P_4)$.

5 Conclusions

In this paper we have proved that in the case of the family $\Psi(G)$, the accessibility property implies the exchange property, and the resulting greedoids form a proper subfamily of the class of interval greedoids. The graphs, whose families of local maximum stable sets are either antimatroids or matroids, have been described completely.

Open problem: characterize the interval greedoids, the matroids, and the antimatroids produced by $\Psi(G)$.

References

[1] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, in: Graph Theory and Algorithms, Lecture Notes in Computer Science **108** (1980) 108-123, Springer-Verlag, Berlin.

[2] A. Björner, G. M. Ziegler, *Introduction to greedoids*, in N. White (ed.), *Matroid Applications*, 284-357, Cambridge University Press, 1992.

[3] G. H. Cheston, E. O. Hare, S. T. Hedetniemi, R. C. Laskar, *Simplicial graphs*, Congressus Numerantium 67 (1988) 105-113.

[4] R. W. Deming, *Independence numbers of graphs - an extension of the König–Egerváry theorem*, Discrete Mathematics **27** (1979) 23–33.

[5] E. Egervary, *On combinatorial properties of matrices*, Mat. Lapok **38** (1931) 16-28.

[6] O. Favaron, *Very well-covered graphs*, Discrete Mathematics **42** (1982) 177-187.

[7] M. C. Golumbic, T. Hirst, M. Lewenstein, *Uniquely restricted matchings*, Algorithmica **31** (2001) 139-154.

[8] G. Gunther, B. Hartnell, D.F. Rall, *Graphs whose vertex independence number is unaffected by single edge addition or deletion*, Discrete Applied Mathematics **46** (1993) 167–172.

[9] G. Hopkins, W. Staton, *Graphs with unique maximum independent sets*, Discrete Mathematics **57** (1985) 245-251.

[10] B. Korte, L. Lovász, R. Schrader, *Greedoids*, Springer-Verlag, Berlin, 1991.

[11] D. König, *Graphen und Matrizen*, Mat. Lapok **38** (1931) 116-119.

[12] V. E. Levit, E. Mandrescu, *On the structure of α-stable graphs*, Discrete Mathematics **236** (2001) 227-243.

[13] V. E. Levit, E. Mandrescu, *Unicycle bipartite graphs with only uniquely restricted maximum matchings*, Proceedings of the Third International Conference on Combinatorics, Computability and Logic, (DMTCS'1), Springer, (C. S. Calude, M. J. Dinneen and S. Sburlan eds.) (2001) 151-158.

[14] V. E. Levit, E. Mandrescu, *A new greedoid: the family of local maximum stable sets of a forest*, Discrete Applied Mathematics **124** (2002) 91-101.

[15] V. E. Levit, E. Mandrescu, *Local maximum stable sets in bipartite graphs with uniquely restricted maximum matchings*, Discrete Applied Mathematics **132** (2004) 163-174.

[16] V. E. Levit, E. Mandrescu, *Triangle-free graphs with uniquely restricted maximum matchings and their corresponding greedoids*, Discrete Applied Mathematics **155** (2007) 2414–2425.

[17] V. E. Levit, E. Mandrescu, *On local maximum stable sets of the corona of a path with complete graphs*, Proceedings of the 6^{th} Congress of Romanian Mathematicians, University of Bucharest, Romania (2007) (in press).

[18] V. E. Levit, E. Mandrescu, *Well-covered graphs and greedoids*, Proceedings of the 14^{th} Computing: The Australasian Theory Symposium (CATS2008), Wollongong, NSW, Conferences in Research and Practice in Information Technology Volume **77** (2008) 89-94.

[19] V. E. Levit, E. Mandrescu, *The clique corona operation and greedoids*, Combinatorial Optimization and Applications, Second International Conference, COCOA 2008, Lecture Notes in Computer Science **5165** (2008) 384-392.

[20] V. E. Levit, E. Mandrescu, *Graph operations that are good for greedoids*, MOPTA 2008, University of Guelph, Guelph, Canada, prE-print arXiv:0809.1806v1 (2008) 9 pp.

[21] G. L. Nemhauser, L. E. Trotter, Jr., *Vertex packings: structural properties and algorithms*, Mathematical Programming **8** (1975) 232-248.

[22] M. D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory **8** (1970) 91-98.

[23] W. Siemes, J. Topp, L. Volkman, *On unique independent sets in graphs*, Discrete Mathematics **131** (1994) 279-285.

[24] F. Sterboul, *A characterization of the graphs in which the transversal number equals the matching number*, Journal of Combinatorial Theory Series B **27** (1979) 228–229.

[25] J. Zito, *The structure and maximum number of maximum independent sets in trees*, Journal of Graph Theory **15** (1991) 207-221.